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Asymptotic Behavior of Nonoscillatory Solutions of Neutral Difference Equations

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Abstract—The authors consider the m^{th}-order neutral difference equation

$$D_m(y(n)+p(n)y(n-k))+q(n)f(y(\sigma(n)))=e(n),$$

where $m \geq 1$, $\{p(n)\}$, $\{q(n)\}$, $\{e(n)\}$, and $\{a_1(n)\},\{a_2(n)\},\dots,\{a_{m-1}(n)\}$ are real sequences, $a_i(n)>0$ for $i=1,2,\dots,m-1$, $a_m(n)\equiv 1$, $D_0z(n)=y(n)+p(n)y(n-k)$, $D_iz(n)=a_i(n)\Delta D_{i-1}z(n)$ for $i=1,2,\dots,m$, k is a positive integer, $\{\sigma(n)\}\to\infty$ is a sequence of positive integers, and $f:\mathbb{R}\to\mathbb{R}$ is continuous with $uf(u)>0$ for $u\neq 0$. In the case where $\{q(n)\}$ is allowed to oscillate, they obtain sufficient conditions for all bounded nonoscillatory solutions to converge to zero, and if $\{q(n)\}$ is a nonnegative sequence, they establish sufficient conditions for all nonoscillatory solutions to converge to zero. Examples illustrating the results are included throughout the paper.

Keywords—Asymptotic behavior, Difference equations, Higher order, Neutral equations, Nonoscillatory solutions.

1. INTRODUCTION

Consider the m^{th}-order neutral difference equation

$$D_m(y(n)+p(n)y(n-k))+q(n)f(y(\sigma(n)))=e(n), \tag{E}$$

where $m\geq 1$, $n\in\mathbb{N}=\{0,1,2,\dots\}$, $\{p(n)\}$, $\{q(n)\}$, $\{e(n)\}$, and $\{a_1(n)\},\{a_2(n)\},\dots,\{a_{m-1}(n)\}$ are real sequences, $a_i(n)>0$ for $i=1,2,\dots,m-1$ and all $n\in\mathbb{N}$, $a_m(n)\equiv 1$, $D_0z(n)=y(n)+p(n)y(n-k)$, $D_iz(n)=a_i(n)\Delta D_{i-1}z(n)$ for $i=1,2,\dots,m$, k is a positive integer, $\{\sigma(n)\}$ is a sequence of positive integers with $\sigma(n)\to\infty$ as $n\to\infty$, and $f:\mathbb{R}\to\mathbb{R}$ is continuous with $uf(u)>0$ for $u\neq 0$. We let $\theta=\max\{\min_{n\in\mathbb{N}}\sigma(n),k\}$ and define a *solution* of equation (E) to be a nontrivial real sequence $\{y(n)\}$ defined for $n\geq N_0-\theta$, $N_0\in\mathbb{N}$, and satisfying (E) for

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$n \geq N_0$. Such a solution is said to be *oscillatory* if for every $N \in \mathbb{N}$ there exist $n_1, n_2 \in \mathbb{N}$ with $n_2 > n_1 > N$ and $y(n_1)y(n_2) \leq 0$, and it is said to be *nonoscillatory* otherwise.

The oscillatory properties of solutions of neutral difference equations with $\{q(n)\}$ eventually nonnegative or nonpositive have been studied by many authors; the recent monographs by Agarwal [1], Agarwal, Grace and O'Regan [2], and Agarwal and Wong [3] survey much that is known and contain extensive bibliographies on such problems. However, relatively few results have been obtained for the case where $\{q(n)\}$ is allowed to change signs; as recent contributions, we cite the papers of Agarwal, Manuel and Thandapani [4,5], and Zafer [6,7]. Our aim in this paper is to study the asymptotic behavior of the nonoscillatory solutions of (E) in this case. In addition, we are considering equations with forcing terms which also sets our results apart from many others. In particular, if $\{q(n)\}$ is allowed to oscillate, we obtain sufficient conditions for all bounded solutions of (E) to converge to zero as $n \to \infty$. In addition, if $\{q(n)\}$ is a nonnegative sequence, we establish sufficient conditions for all nonoscillatory solutions of (E) to approach zero as $n \to \infty$. Examples to illustrate our results are included throughout the paper.

2. PRELIMINARIES

In this section, we present some lemmas that are needed in the proofs of our main results. We introduce the notation

$$P_0(n) = 1, \quad P_{j+1}(n) = P_j(n)p(n - jk), \qquad j = 0, 1, 2, \dots. \tag{1}$$

LEMMA 1. *Suppose there exist constants p_1, $p_2 \in \mathbb{R}$ such that*

$$|p(n)| \leq p_1 < 1 \qquad \text{and} \qquad p(n)p(n-k) \geq 0 \tag{2}$$

or

$$p_2 \leq p(n) \leq 0,$$

for $n \geq n_0 \in \mathbb{N}_0$. If $y(n) > 0$, $\liminf_{n\to\infty} y(n) = 0$, and $\lim_{n\to\infty} z(n) = L$ for some $L \in \mathbb{R}$, then $L = 0$.

The proof of the above lemma is similar to the proof of Lemma 1 in [8] and will be omitted.

LEMMA 2. *Suppose that either (2) holds or there is a constant $p_3 \in \mathbb{R}$ such that*

$$p(n) \leq p_3 < -1, \tag{3}$$

for $n \geq n_0 \in \mathbb{N}_0$. If $0 \leq y(n) \leq y_0 < \infty$ and $\lim_{n\to\infty} z(n) = 0$, then $\lim_{n\to\infty} y(n) = 0$.

PROOF. The proof when case (2) holds is similar to the proof of Lemma 2 in [9] and is omitted. Now suppose (3) holds; then

$$y(n) \leq \frac{1}{p_3}\left[z(n+k) - y(n+k)\right],$$

for $n \geq n_0$. Iterating, we have

$$y(n) \leq \frac{1}{p_3}z(n+k) - \frac{1}{p_3^2}z(n+2k) + \cdots + (-1)^{t-1}\frac{1}{p_3^t}z(n+tk) + (-1)^t\frac{1}{p_3^t}y(n+tk),$$

for $n \geq n_0$. Since $\lim_{n\to\infty} z(n) = 0$, for any $\epsilon > 0$ there is an $n_1 \geq n_0$ such that $|z(n)| < ((|p_3| - 1)/2)\epsilon$ for $n \geq n_1$. Also, we can choose a positive integer t_0 such that $y_0/|p_3|^t < \epsilon/2$ for $t \geq t_0$. Then, $y(n) < \epsilon$ for $n > \max\{n_1, t_0\}$, so $\lim_{n\to\infty} y(n) = 0$.

LEMMA 3. *Suppose $|p(n)| \le p_1 < 1$. If $y(n) > 0$ and $\{z(n)\}$ is bounded from above ($y(n) < 0$ and $\{z(n)\}$ is bounded from below) for $n \ge n_1 \ge n_0$, then $\{y(n)\}$ is bounded.*

PROOF. Suppose $y(n) > 0$ and $z(n) \le K < \infty$ for $n \ge n_1$. Then

$$y(n) = z(n) - p(n)y(n-k) = z(n) - P_1(n)z(n-k) + P_2(n)y(n-2k),$$

for $n \ge n_0 + 2k$. It follows that

$$y(n) = \sum_{j=0}^{t-1}(-1)^j P_j(n)z(n-jk) + (-1)^t P_t(n)y(n-tk), \tag{4}$$

for $n \ge n_0 + tk$, $t = 1, 2, \dots$.

Let $0 < y(n) \le C$ for $n_0 \le n \le n_0 + k$; then $y(n-tk) \le C$ for $n_0 + tk \le n \le n_0 + (t+1)k$. Now since $|p(n)| \le p_1 < 1$, it follows that $|P_j(n)| \le p_1^j < 1$ for $n \ge n_0 + jk$, $j = 1, 2, \dots$. From (4), we then have

$$0 < y(n) \le K \sum_{j=0}^{t-1} p_1^j + p_1^t C \le \frac{K}{1-p_1} + C = K_1 < \infty,$$

for $n_0 + tk \le n \le n_0 + (t+1)k$, $t = 1, 2, \dots$. This implies $0 < y(n) \le K_1$ for $n \ge n_0$. The proof in case $y(n) < 0$ and $\{z(n)\}$ is bounded from below is similar.

Next, we set $A_0(n) = 1$ and let

$$A_j(n) = \sum_{s=n_0}^{n} \frac{A_{j-1}(s)}{a_j(s)}, \qquad \text{if } A_j(\infty) = \infty, \quad \text{for } j = 1, 2, \dots, m-1, \tag{5}$$

and

$$A_j(n) = \sum_{s=n+1}^{\infty} \frac{A_{j-1}(s)}{a_j(s)}, \qquad \text{if } A_j(n_0) < \infty, \quad \text{for } j = 1, 2, \dots, m-1. \tag{6}$$

LEMMA 4. *Suppose $|p(n)| \le p_1 < 1$, and for $j = 1, 2, \dots, m$ and $N \ge n_0$, let*

$$u_j(n) = \sum_{s=N+1}^{n} A_{j-1}(s)\Delta D_{j-1}z(s).$$

(i) *If $\lim_{n\to\infty} u_j(n) = \infty$ $(-\infty)$ for some $j = 2, 3, \dots, m$, then $\lim_{n\to\infty} u_i(n) = \infty$ $(-\infty)$ for $i = 1, 2, \dots, j-1$.*
(ii) *If $\{z(n)\}$ is a bounded sequence and $\lim_{n\to\infty} u_m(n)$ exists, then $\lim_{n\to\infty} z(n) = z_0 \in \mathbb{R}$.*
(iii) *If (5) holds, then $\lim_{n\to\infty} D_i z(n) = 0$ for $i = 1, 2, \dots, m-1$.*

PROOF. It is easy to see that for each $j = 1, 2, \dots, m-1$, the sequence $\{u_j(n)\}$ satisfies the difference equation

$$\frac{A_j(n)}{\Delta A_j(n)} \Delta u_j(n) - u_j(n) = \epsilon v_{j+1}(n), \qquad n \ge N, \tag{7_j}$$

where

$$v_{j+1}(n) = u_{j+1}(n) + A_j(N+1)D_j z(N+1) - \epsilon A_{j-1}(N+1)\Delta D_{j-1}z(N+1), \qquad j = 1, 2, \dots, m-1,$$

and $\epsilon = +1$ or -1 depending on whether (5) or (6) holds, respectively. Now if (5) (respectively, (6)) holds, then $A_j(n) > 0$, $\Delta A_j(n) > 0$, and $\lim_{n\to\infty} A_j(n) = \infty$ ($A_j(n) > 0$, $\Delta A_j(n) < 0$, and $\lim_{n\to\infty} A_j(n) = 0$) for $j = 1, 2, \dots, m-1$ and $n \ge N$. From (7_j), we have

$$u_j(n) = \epsilon A_j(n) \sum_{s=N}^{n-1} \frac{\Delta A_j(s)}{A_j(s)A_j(s+1)} v_{j+1}(s), \qquad j = 1, 2, \dots, m-1. \tag{8_j}$$

(i) Let $j \in \{1, 2, \dots, m-1\}$ with $\lim_{n\to\infty} u_{j+1}(n) = \infty$; then $\lim_{n\to\infty} v_{j+1}(n) = \infty$. By (8_j) and either (5) or (6), we have $\lim_{n\to\infty} u_j(n) = \infty$. If $j > 1$, we can repeat this process to obtain successively that $\lim_{n\to\infty} u_i(n) = \infty$ for $i = j-1, j-2, \dots, 1$.

(ii) Let $\{z(n)\}$ be bounded. Then $u_1(n) = z(n+1) - z(N+1)$ is also bounded. If $\lim_{n\to\infty} u_m(n)$ exists, then by (7_{m-1}), $\lim_{n\to\infty} u_{m-1}(n)$ exists. Proceeding successively, we obtain that $\lim_{n\to\infty} u_j(n)$ exists for each $j = 1, 2, \dots, m-2$. By Part (i) and the fact that $\{u_1(n)\}$ is bounded, there exist finite numbers b_j, $j = 1, 2, \dots, m$ such that $\lim_{n\to\infty} u_j(n) = b_j$, $j = 1, 2, \dots, m$. Therefore, from (7_j), we have

$$\begin{aligned}&\lim_{n\to\infty}\left[\frac{A_j(n)}{\Delta A_j(n)}\Delta u_j(n) - u_j(n)\right]\\ &= \epsilon\left[b_{j+1} + A_j(N+1)D_j z(N+1) - \epsilon A_{j-1}(N+1)\Delta D_{j-1}(N+1)\right] = c_{j+1},\end{aligned} \tag{9_j}$$

for $j = 1, 2, \dots, m$. Now from (9_j) and the definition of $\{u_j(n)\}$, we have

$$\lim_{n\to\infty}\frac{A_j(n)}{\Delta A_j(n)}\Delta u_j(n) = \lim_{n\to\infty} A_j(n) D_j z(n+1) = c_{j+1} - b_j = d_j \tag{10}$$

and

$$\lim_{n\to\infty} z(n) = b_1 + z(N) = c_1.$$

(iii) If $A_j(\infty) = \infty$ for $j = 1, 2, \dots, m-1$, then (10) yields

$$\lim_{n\to\infty} D_j z(n) = 0, \qquad j = 1, 2, \dots, m-1.$$

3. MAIN RESULTS

We will make use of the notation $q^+(n) = \max\{q(n), 0\}$ and $q^-(n) = \max\{-q(n), 0\}$ so that $q(n) = q^+(n) - q^-(n)$ for all $n \geq n_0$.

THEOREM 1. *Suppose that either condition (2) holds, or there exist constants p_4 and p_5 such that*

$$p_4 \leq p(n) \leq p_5 < -1, \tag{11}$$

for $n \geq n_0$. In addition, assume that

$$\sum_{n=n_0}^{\infty} A_{m-1}(n)|e(n)| < \infty, \tag{12}$$

and either

$$\sum_{n=n_0}^{\infty} A_{m-1}(n) q^+(n) = \infty \qquad \text{and} \qquad \sum_{n=n_0}^{\infty} A_{m-1}(n) q^-(n) < \infty \tag{13}$$

or

$$\sum_{n=n_0}^{\infty} A_{m-1}(n) q^+(n) < \infty \qquad \text{and} \qquad \sum_{n=n_0}^{\infty} A_{m-1}(n) q^-(n) = \infty. \tag{14}$$

Then every bounded solution $\{y(n)\}$ of (E) is either oscillatory or satisfies $\lim_{n\to\infty} y(n) = \lim_{n\to\infty} z(n) = 0$. In addition, if (5) holds, then

$$\lim_{n\to\infty} D_i z(n) = 0, \qquad \text{for } i = 1, 2, \dots, m-1. \tag{15}$$

PROOF. Let $\{y(n)\}$ be a bounded nonoscillatory solution of (E), say $y(n) > 0$, $y(n-k) > 0$, and $y(\sigma(n)) > 0$ for $n \geq N$. Since $\{y(n)\}$ is bounded, (2) or (11) implies $\{z(n)\}$ is bounded. Multiplying (E) by A_{m-1} and summing from $N+1$ to n, we have

$$\begin{aligned} u_m(n) = \sum_{s=N+1}^{n} A_{m-1}(s) D_m z(s) &= \sum_{s=N+1}^{n} A_{m-1}(s) q^-(s) f(y(\sigma(s))) \\ &- \sum_{s=N+1}^{n} A_{m-1}(s) q^+(s) f(y(\sigma(s))) + \sum_{s=N+1}^{n} A_{m-1}(s) e(s). \end{aligned} \tag{16}$$

Suppose (13) holds. If

$$\sum_{n=N+1}^{n} A_{m-1}(n) q^+(n) f(y(\sigma(n))) = \infty,$$

then the boundedness of $\{y(n)\}$, (12), (13), and (16) imply $\lim_{n\to\infty} u_m(n) = -\infty$. By Lemma 4(i), $\lim_{n\to\infty} z(n) = -\infty$ which contradicts the boundedness of $\{z(n)\}$. Hence,

$$\sum_{n=N+1}^{n} A_{m-1}(n) q^+(n) f(y(\sigma(n))) < \infty. \tag{17}$$

Letting $n \to \infty$ in (16), we have $\lim_{n\to\infty} u_m(n) = b_m$ for some $b_m \in \mathbb{R}$. Lemma 4(ii) and the boundedness of $\{z(n)\}$ imply $\lim_{n\to\infty} z(n) = b_0 \in \mathbb{R}$. From (13) and (17), we have $\liminf_{n\to\infty} y(n) = 0$, so by Lemma 1, $\lim_{n\to\infty} z(n) = 0$. Lemma 2 then implies $\lim_{n\to\infty} y(n) = 0$.

Finally, if (5) also holds, then we have $\lim_{n\to\infty} D_j z(n) = 0$, $j = 1, 2, \dots, m-1$. The proof if (14) holds is similar.

EXAMPLE 1. Consider the difference equation

$$\Delta\left(2^{-n}\Delta\left(y(n) - \frac{1}{8}y(n-2)\right)\right) + \frac{2^{-n}}{4}\frac{1+(-1)^n}{2+(-1)^n}y(n-1) = \frac{7}{2^{2n+4}}(2-(-1)^n), \qquad n \geq 3. \tag{E$_1$}$$

Conditions (2), (5), (12), and (13) of Theorem 1 hold. Equation (E$_1$) has the nonoscillatory solution $\{y(n)\} = \{(2-(-1)^n)/2^n\}$. Here $\{z(n)\} = \{(2-(-1)^n)/2^{n+1}\}$, and it is easy to see that $\lim_{n\to\infty} y(n) = \lim_{n\to\infty} z(n) = \lim_{n\to\infty} D_1 z(n) = 0$.

EXAMPLE 2. Consider the difference equation

$$\Delta\left(2^{2n}\Delta\left(y(n) - \frac{3}{4}y(n-2)\right)\right) - 2^{2n+1}\left(n^2+1\right)y(n+1) = -n^2 2^n, \qquad n \geq 3. \tag{E$_2$}$$

Conditions (2), (6), (12), and (14) of Theorem 1 are satisfied, and equation (E$_2$) has the nonoscillatory solution $\{y(n)\} = \{1/2^n\}$. For this equation, $\{z(n)\} = \{-1/2^{n-1}\}$, $\{D_1 z(n)\} = \{2^n\}$, $\lim_{n\to\infty} y(n) = \lim_{n\to\infty} z(n) = 0$, and $\lim_{n\to\infty} D_1 z(n) = \infty$. We see that if (5) is not satisfied, then (15) need not hold.

As an example with $\{p(n)\}$ and $\{q(n)\}$ oscillatory, we have the following example.

EXAMPLE 3. Consider the equation

$$\begin{aligned} \Delta^3\left(2^{n+1}\Delta\left(y(n) + \frac{(-1)^n}{4}y(n-2)\right)\right) &+ \left[2^{n-1}\left(1+(-1)^n\right) - \frac{1+(-1)^{n+1}}{2^n n^2}\right]y(n-1) \\ &= (1+25(-1)^n) - \frac{1+(-1)^{n+1}}{2^{2n-1}n^2}, \qquad n \geq 3. \end{aligned} \tag{E$_3$}$$

Hypotheses (2), (6), (12), and (13) of Theorem 1 are satisfied. Equation (E$_3$) has the nonoscillatory solution $\{y(n)\} = \{1/2^n\}$, and we see that $\{z(n)\} = \{(1+(-1)^n)/2^n\}$. Here, $\lim_{n\to\infty} y(n) = \lim_{n\to\infty} z(n) = 0$ but $D_j z(n) \not\to 0$ as $n \to \infty$ for $j = 1, 2, 3$.

The following two examples show that in some sense the conclusion of Theorem 1 is the best possible under the given hypotheses.

EXAMPLE 4. As a simple example of an equation that satisfies all the conditions of Theorem 1 and has an unbounded oscillatory solution, consider

$$\Delta\left(2^{-n}\Delta\left(y(n)-\frac{1}{2}y(n-2)\right)\right)+21(2^{-n})y(n-2)=0, \qquad n\geq 2. \tag{E$_4$}$$

Here, $\{y(n)\}=\{(-2)^n\}$ is such a solution.

EXAMPLE 5. Consider the equation

$$\Delta\left(2^{-n}\Delta\left(y(n)+ay(n-b)\right)\right)+\frac{3(1+a)}{2^n}y^3(n-c)=0, \qquad n\geq b+c, \tag{E$_5$}$$

where b and c are positive integers and a is a constant with either $a<-1$ or $-1<a<1$. The hypotheses of Theorem 1 are satisfied and this equation has the bounded oscillatory solution $\{y(n)\}=\{(-1)^n\}$ that does not converge to zero.

In our next theorem, we ask that $\{q(n)\}$ be nonnegative and obtain that any solution is either oscillatory or converges to zero as $n\to\infty$.

THEOREM 2. *Assume that conditions (2), (6), and (12) hold, $q(n)\geq 0$, and*

$$\sum_{n=n_0}^{\infty}A_{m-1}(n)q(n)=\infty.$$

Then any solution $\{y(n)\}$ of equation (E) is either oscillatory or satisfies $\lim_{n\to\infty}y(n)=0$ and $\lim_{n\to\infty}z(n)=0$.

PROOF. Let $\{y(n)\}$ be a bounded nonoscillatory solution of (E), say $y(n)>0$, $y(n-k)>0$, and $y(\sigma(n))>0$ for $n\geq N$. Multiplying (E) by A_{m-1} and summing from $N+1$ to n, we obtain

$$u_m(n)=\sum_{s=N+1}^{n}A_{m-1}(s)D_mz(s)=\sum_{s=N+1}^{n}A_{m-1}(s)e(s)-\sum_{s=N+1}^{n}A_{m-1}(s)q(s)f(y(\sigma(s))).$$

Similar to the argument used in the proof of Theorem 1, the above expression implies $\{u_m(n)\}$ is bounded from above, i.e.,

$$\hat{u}_m(n)=u_m(n)+A_{m-1}(N+1)D_{m-1}z(N+1)+A_{m-2}(N+1)\Delta D_{m-2}z(N+1)<K<\infty,$$

for some $K>0$ and all $n\geq N$. From (6) and (8_{m-1}), we obtain

$$u_{m-1}(n)\leq -KA_{m-1}(n)\sum_{s=N}^{n-1}\frac{\Delta A_{m-1}(s)}{A_{m-1}(s)A_{m-1}(s+1)}=K\left[1-\frac{A_{m-1}(n)}{A_{m-1}(N)}\right]\leq K,$$

for $n\geq N$. Repeating this argument $m-2$ times, we eventually obtain that $u_1(n)=z(n+1)-z(N+1)$ is bounded from above. By Lemma 3, $\{y(n)\}$ is bounded, so applying Theorem 1, we have $\lim_{n\to\infty}y(n)=\lim_{n\to\infty}z(n)=0$.

EXAMPLE 6. Consider the difference equation

$$\begin{gathered}\Delta\left(n(n+1)\Delta\left((n+2)(n+3)\Delta\left(n(n+1)\Delta\left(y(n)-\frac{n-1}{2(n+1)}y(n-1)\right)\right)\right)\right)\\ +n^4y^3(\gamma n)=\frac{n}{\gamma^3(\gamma n+1)^3}, \qquad n\geq 2,\end{gathered} \tag{E$_6$}$$

where γ is a positive integer. All conditions of Theorem 2 are satisfied, so every nonoscillatory solution of (E_6) tends to zero as $n\to\infty$; $\{y(n)\}=\{1/n(n+1)\}$ is such a solution of this equation.

In our final result, we examine equation (E) in the case where $\{p(n)\}$ is bounded, and we establish criteria for all bounded nonoscillatory solutions to tend to zero as $n\to\infty$. Here, we again allow the sequence $\{q(n)\}$ to change signs.

THEOREM 3. *Assume that* $\{p(n)\}$ *is bounded,*

$$\sum_{n=n_0}^{\infty} \frac{1}{a_i(n)} = \infty, \qquad i = 1, 2, \ldots, m-1, \tag{18}$$

and

$$\sum_{n=n_0}^{\infty} |e(n)| < \infty. \tag{19}$$

If either

$$\sum_{n=n_0}^{\infty} q^+(n) = \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} q^-(n) < \infty \tag{20}$$

or

$$\sum_{n=n_0}^{\infty} q^+(n) < \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} q^-(n) = \infty, \tag{21}$$

then every bounded solution $\{y(n)\}$ *of (E) is either oscillatory or satisfies* $\liminf_{n\to\infty} |y(n)| = 0$ *and* $\lim_{n\to\infty} D_i z(n) = 0$ *for* $i = 0, 1, \ldots, m-1$.

PROOF. Let $\{y(n)\}$ be a bounded nonoscillatory solution of (E) with $y(n) > 0$, $y(n-k) > 0$, and $y(\sigma(n)) > 0$ for $n \geq N$. Since $\{p(n)\}$ and $\{y(n)\}$ are bounded, $\{z(n)\}$ is bounded. Summing (E) from N to $n-1$, we obtain

$$D_{m-1}z(n) - D_{m-1}z(N) + \sum_{s=N}^{n-1} q^+(s) f(y(\sigma(s))) = \sum_{s=N}^{n-1} e(s) + \sum_{s=N}^{n-1} q^-(s) f(y(\sigma(s))).$$

Suppose (20) holds. If

$$\sum_{s=N}^{\infty} q^+(s) f(y(\sigma(s))) = \infty,$$

then $\lim_{n\to\infty} D_{m-1}z(n) = -\infty$. Condition (18) then implies $\lim_{n\to\infty} z(n) = -\infty$, which contradicts the boundedness of $\{z(n)\}$. Hence,

$$\sum_{s=N}^{\infty} q^+(s) f(y(\sigma(s))) < \infty, \tag{22}$$

from which it immediately follows that $\liminf_{n\to\infty} y(n) = 0$. Now, (19), (20), and (22) imply $\lim_{n\to\infty} D_{m-1}z(n)$ exists and is finite. In view of (18) and the boundedness of $\{z(n)\}$, we see that $\lim_{n\to\infty} D_j z(n) = 0$ for $j = 1, 2, \ldots m-1$. A similar proof handles the case where (21) holds.

We conclude this paper with the following example.

EXAMPLE 7. The difference equation

$$\begin{gathered} \Delta\left((n+1)\Delta\left((n+1)\Delta\left(y(n) + \frac{n-1}{n} y(n-1)\right)\right)\right) \\ + \frac{1}{n+1} y^{1/3}(n+1) = \frac{-2}{n(n+1)} + \frac{1}{(n+1)^{4/3}}, \qquad n \geq 2, \end{gathered} \tag{E_7}$$

satisfies the hypotheses of Theorem 3. This equation has the nonoscillatory solution $\{y(n)\} = \{1/n\}$. Here we have $\{z(n)\} = \{2/n\}$, and it is easy to see that $\liminf_{n\to\infty} y(n) = 0$ and $\lim_{n\to\infty} D_i z(n) = 0$ for $i = 0, 1, 2$.

REMARK. The results in this paper extend some of those obtained in [10]. In addition to the fact that the equations under consideration here are of the "neutral type" and those in [10] are not, in [10] we only considered the case where condition (6) above holds.

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